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Closed-form absorption probability of certain $D = 5$ and $D = 4$ black holes and leading-order cross-section of generic extremal p-branes

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Abstract: We obtain the closed-form absorption probabilities for minimally-coupled massless scalars propagating in the background of $D = 5$ single-charge and $D = 4$ two-charge black holes. These are the only two examples of extremal black holes with non-vanishing absorption probabilities that can be solved in closed form for arbitrary incident frequencies. In both cases, the absorption probability vanishes when the frequency is below a certain threshold, and we discuss the connection between this phenomenon and the behaviour of geodesics in these black hole backgrounds. We also obtain leading-order absorption cross-sections for generic extremal p-branes, and show that the expression for the cross-section as a function of frequency coincides with the leading-order dependence of the entropy on the temperature in the corresponding near-extremal p-branes.

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Contents

1. Introduction

There has been considerable interest recently in studying absorption probabilities for fields propagating in various black hole and p-brane backgrounds [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. One of the motivations is inspired by the conjectured duality of supergravity on an AdS spacetime and the conformal field theory on the boundary of the AdS [17]. To obtain the absorption probability is however by itself a subject of interest. The wave equation for a minimally-coupled scalar in an extremal p-brane background depends only on the metric of the p-brane, which has the form

$$ds^2 = \prod_{\alpha=1}^{N} H_\alpha^{-\frac{\tilde{d}}{D-2}}\, dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^{N} H_\alpha^{\frac{d}{D-2}}\, dy^m dy^m\,, \tag{1.1}$$

where $d = p + 1$ is the dimension of the world volume of the p-brane and $\tilde{d} = D - d - 2$, and $H_\alpha = 1 + Q_\alpha/r^{\tilde{d}}$ are harmonic functions in the transverse space y^m, with $r^2 = y^m y^m$. It follows that the wave equation $\partial_M(\sqrt{g}\, g^{MN}\partial_N \Phi) = 0$ for the massless minimally-coupled scalar, with the ansatz $\Phi(t, r, \theta_i) = \phi(r)\, Y(\theta_i)\, e^{-i\omega t}$, takes the following form:

$$\frac{d^2\phi}{d\rho^2} + \frac{\tilde{d}+1}{\rho}\frac{d\phi}{d\rho} + \left[\prod_{\alpha=1}^{N}\left(1 + \frac{\lambda_\alpha^{\tilde{d}}}{\rho^{\tilde{d}}}\right) - \frac{\ell(\ell+\tilde{d})}{\rho^2}\right]\phi = 0\,, \tag{1.2}$$

where $\rho = \omega\, r$ and $\lambda_\alpha = \omega\, Q_\alpha^{1/\tilde{d}}$. Note that the wave equation depends on $\tilde{d}$, but is independent of the world-volume dimension d. This implies that the wave equation is invariant under double dimensional reduction of the corresponding p-brane. It is not, however, invariant under vertical dimensional reduction, where $\tilde{d}$ rather than d is reduced.

In general, the wave equation (1.2) is not exactly solvable; it cannot be mapped into any known second-order differential equation. The leading-order absorption probability can be obtained by matching approximate solutions of the wave equation that are valid in overlapping inner and outer regions; this technique was pioneered in absorption probability calculations for the Schwarzschild black hole [18]. Only for a few examples are the wave equations exactly solvable, including the extremal D3-brane [13] and the extremal dyonic string [16], corresponding to $N = 1$, $\tilde{d} = 4$ and $N = 2$, $\tilde{d} = 2$ respectively. In these two cases, the wave equation (1.2) can be cast into the form of the modified Mathieu equation [13, 16], which has been studied in great detail in the mathematical literature [19]. However, even in these two cases the final results for the absorption probabilities can only be expressed in terms of power-series expansions in some small parameter, which depends on the frequency of the wave and, in the case of dyonic string, on the relative ratio of the electric and magnetic charges as well.

In this paper, we shall consider two further examples, where the absorption probability can in fact be obtained in closed form for scalar waves of arbitrary frequencies. One example is the extremal single-charge black hole in $D = 5$, and the other is the extremal two-charge black hole in $D = 4$. We discuss these two examples in sections 2 and 3 respectively. In both cases, an interesting phenomenon occurs, namely that the absorption probability vanishes below a certain threshold frequency for the incident scalar wave. In section 4, we observe that this is related to special features of the behaviour of infalling geodesics in these two black holes.

Although the wave equation (1.2) cannot be solved in general, the leading-order absorption probability for low energy waves can nevertheless be obtained by matching inner and outer solutions of the wave equations. We shall apply this technique in section 5, and obtain the leading-order absorption for general extremal p-branes, which we find to be given by

$$\sigma = c\,(Q_1\,Q_2\,\cdots\,Q_N)^{\frac{\tilde{d}}{N\tilde{d}-2}}\,\omega^{\frac{2\tilde{d}}{N\tilde{d}-2}-1}\,, \tag{1.3}$$

where Q_α are the charges and c is a charge-independent numerical constant. This result coincides with the leading-order dependence of the entropy on the temperature of the corresponding near-extremal p-brane, namely

$$S = \tilde{c}\,(Q_1\,Q_2\,\cdots\,Q_N)^{\frac{\tilde{d}}{N\tilde{d}-2}}\,T^{\frac{2\tilde{d}}{N\tilde{d}-2}-1}\,, \tag{1.4}$$

which we obtain in section 6. Such a parallel implies a connection between the entropy of near-extremal p-branes and the low-energy absorption by the corresponding extremal p-branes.

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2. Single-charge black hole in $D = 5$

In maximal $D = 5$ supergravity, each of the 27 vector potentials can support a single-charge black hole. These black holes form a 27-dimensional representation under the Weyl group of E_6 [20]. A complete classification of the field configurations that supports N-charge p-branes in maximal supergravities can be found in [21]. The single-charge black hole in $D = 5$ corresponds to $N = 1$, $\tilde{d} = 2$, and hence from (1.2) the scalar wave equation is

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} + \left[1 + \frac{\lambda^2 - \ell(\ell+2)}{\rho^2}\right]\phi = 0\,. \tag{2.1}$$

This can be solved exactly, giving

$$\phi = \frac{\alpha}{\rho}\,J_{\mathrm{i}q}(\rho) + \frac{\beta}{\rho}\,J_{-\mathrm{i}q}(\rho)\,, \tag{2.2}$$

where we have defined

$$q \equiv \sqrt{\lambda^2 - (\ell+1)^2}\,. \tag{2.3}$$

We see from this that the behaviour of the solutions will be radically different depending on whether λ is larger than $(\ell+1)$ or smaller than $(\ell+1)$. In particular, it will turn out that we only have wave-like behaviour near the horizon if $\lambda > \ell+1$. (It is always understood in our discussions that λ is, without loss of generality, taken to be non-negative.)

Expanding (2.2) around $\rho = 0$, using (A.2), we find that the wave-function near the horizon takes the form

$$\phi \sim \frac{\alpha}{\rho\,\Gamma(1+\mathrm{i}q)}\,e^{\mathrm{i}q\log(\rho/2)} + \frac{\beta}{\rho\,\Gamma(1-\mathrm{i}q)}\,e^{-\mathrm{i}q\log(\rho/2)}\,. \tag{2.4}$$

(We have used the identity $x^y = e^{y\log x}$ here.) The boundary condition at the horizon, which requires that the wave be purely ingoing there, implies that we must have $\alpha = 0$.

Expanding (2.2) at large ρ, using (A.1), we find that the wave-function near infinity takes the form

$$\phi \sim -\frac{\beta}{2\rho}\sqrt{\frac{2}{\pi\rho}}\,e^{\pi q/2}\,e^{\mathrm{i}\pi/4}\Big(-e^{-\mathrm{i}\rho} + \mathrm{i}\,e^{-\pi q}\,e^{\mathrm{i}\rho}\Big)\,. \tag{2.5}$$

Thus comparing with the generic structure $\phi \propto (-e^{-\mathrm{i}\rho} + S_0\,e^{\mathrm{i}\rho})$, we read off the S-matrix:

$$S_0 = \mathrm{i}\,e^{-\pi q}\,. \tag{2.6}$$

From this, it follows that the absorption probability for the single-charge $D = 5$ extremal black hole is given by

$$\begin{aligned} P &= 1 - |S_0|^2 = 1 - e^{-2\pi q} \\ &= 1 - e^{-2\pi\sqrt{\lambda^2-(\ell+1)^2}}\,, \qquad \lambda \ge \ell + 1\,. \end{aligned} \tag{2.7}$$

It should be emphasised that this is an exact result, correct to all orders in λ. It should also be noted that if $\lambda \leq \ell+1$ there is no wave-like behaviour near the horizon, and the absorption probability is therefore zero for all $\lambda \leq \ell+1$.

It is instructive also to compute the absorption probability in this example by taking the ratio of the ingoing fluxes at the horizon and at infinity. The flux in this case is given in general by $F = \mathrm{i}\rho^3\, (\bar{\phi}\, \partial_\rho \phi - \phi\, \partial_\rho \bar{\phi})$, where here ϕ is taken to be purely ingoing component of the scalar wave. Using the near-horizon form of the solution (2.4) (and recalling that the boundary condition required $\alpha = 0$), we see that flux into the horizon is given by

$$F_{\rm horizon} = \frac{2|\beta|^2\, q}{\Big|\Gamma(1-\mathrm{i}q)\Big|^2} = \frac{2|\beta|^2}{\pi}\, \sinh \pi q\,. \tag{2.8}$$

On the other hand the ingoing flux at infinity, calculated using the asymptotic form (2.5), is

$$F_\infty = \frac{|\beta|^2}{\pi}\, e^{\pi q}\,. \tag{2.9}$$

Hence we find that the absorption probability is

$$P = \frac{F_{\rm horizon}}{F_\infty} = 1 - e^{-2\pi q}\,, \tag{2.10}$$

when $\lambda \geq \ell+1$, in agreement with (2.7).

Note that in the non-extremal case the wave equation is more complicated. The absorption probability for non-extremal 3-charge black holes has been calculated for low energies in [3, 5], by the standard techniques involving the overlap between approximate solutions in inner and outer regions. In particular, in [5] two of the three charges were taken to be small in comparison to third, and can be taken to be zero. In this case, when the frequency of the ℓ'th mode is such that $\lambda \leq \ell+1$, the absorption can be seen to be zero in the extremal limit, in agreement with our result above. On the other hand if $\lambda > \ell+1$, the approach in [5] yields an absorption probability for which the extremal limit is singular, and it cannot be compared with the exact result that we have obtained here.

3. Two-charge black hole in $D=4$

Other examples where the wave equation is exactly solvable are for the single-charge and two-charge extremal black holes in four dimensions. The single-charge case gives a wave equation which is precisely equivalent to the familiar Coulomb problem in quantum mechanics, and there is no absorption at any frequency [22]. The two-charge case, on the other hand, is more analogous to the single-charge $D=5$ black hole. In this two-charge case, the wave equation is

$$\frac{d^2\phi}{d\rho^2} + \frac{2}{\rho}\,\frac{d\phi}{d\rho} + \left[\left(1+\frac{\lambda_1}{\rho}\right)\left(1+\frac{\lambda_2}{\rho}\right) - \frac{\ell(\ell+1)}{\rho^2}\right]\phi = 0\,, \tag{3.1}$$

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where we now have two parameters, $\lambda_i = \omega\, R_i = \omega\, Q_i$, associated with the two charges Q_1 and Q_2. (Note that although the two-charge black hole can also be obtained from the intersection of a D1-brane and a D5-brane by dimensional reduction, it requires vertical as well as diagonal reduction steps. Thus the wave equation (3.1) in $D = 4$ is different from the D1-D5 wave equation in $D = 5$, 6 or higher dimensions, which can be cast into the form of the modified Mathiew equation [16].) It is convenient to define the following two constants:

$$p \equiv \frac{1}{2}(\lambda_1 + \lambda_2)\,, \qquad q \equiv \sqrt{4\lambda_1\, \lambda_2 - (2\ell + 1)^2}\,. \tag{3.2}$$

The general solution to (3.1), which is exact, is

$$\begin{aligned} \phi = \alpha\, \rho^{(\mathrm{i}q-1)/2}\, e^{-\mathrm{i}\rho}\, U\left(\frac{1}{2} + \mathrm{i}p + \frac{\mathrm{i}}{2}q, 1 + \mathrm{i}q, 2\mathrm{i}\rho\right) + \\ +\beta\, \rho^{(\mathrm{i}q-1)/2}\, e^{-\mathrm{i}\rho}\, M\left(\frac{1}{2} + \mathrm{i}p + \frac{\mathrm{i}}{2}q, 1 + \mathrm{i}q, 2\mathrm{i}\rho\right), \end{aligned} \tag{3.3}$$

where $U(a, b, z)$ and $M(a, b, z)$ are Kummer's irregular and regular confluent hypergeometric functions, respectively. We give some details of their asymptotic behaviour in the appendix A.

It can be seen from (A.3) and (A.4) that the behaviour of the exact wave function (3.3) near $\rho = 0$, in the vicinity of the horizon, is given by

$$\begin{aligned} \phi \sim\; & \frac{\mathrm{i}\,\alpha\pi\, \rho^{(\mathrm{i}q-1)/2}\, e^{-\mathrm{i}\rho}}{\sinh \pi q} \left\{\frac{1}{\Gamma(\frac{1}{2} + \mathrm{i}p - \frac{\mathrm{i}}{2}q)\,\Gamma(1 + \mathrm{i}q)} - \frac{(2\mathrm{i})^{-\mathrm{i}q}\, \rho^{-\mathrm{i}q}}{\Gamma(\frac{1}{2} + \mathrm{i}p + \frac{\mathrm{i}}{2}q)\,\Gamma(1 - \mathrm{i}q)}\right\} + \\ & +\beta\, \rho^{(\mathrm{i}q-1)/2}\, e^{-\mathrm{i}\rho}\,. \end{aligned} \tag{3.4}$$

Noting that we can write $\rho^{\pm \mathrm{i}q/2}$ as $e^{\pm \mathrm{i}(q/2)\log\rho}$, we see that to satisfy the boundary condition that the wave on the horizon be purely ingoing, the coefficient of $\rho^{\mathrm{i}q/2}$ must vanish. This gives the following relation between α and β:

$$\beta = -\frac{\mathrm{i}\,\alpha\,\pi}{\Gamma(\frac{1}{2} + \mathrm{i}p - \frac{\mathrm{i}}{2}q)\,\Gamma(1 + \mathrm{i}q)\, \sinh \pi q}\,. \tag{3.5}$$

By looking at the large-ρ asymptotic expansions for the Kummer confluent hypergeometric functions, given by (A.5), we find that the wave function near infinity can be written in the form

$$\phi \sim -\frac{\alpha\, e^{\mathrm{i}\,\pi/4}\, 2^{-\mathrm{i}p - \mathrm{i}q/2}\, e^{\pi(p/2 + q/4)}\, (e^{\pi q} + e^{-2\pi p})\, \rho^{-\mathrm{i}p}}{2\sqrt{2}\, \sinh(\pi q)\, \rho} \left(-e^{-\mathrm{i}\rho} + S_0\, e^{\mathrm{i}\rho}\right), \tag{3.6}$$

with the S-matrix S_0 given by

$$S_0 = \frac{2\,\mathrm{i}\,\Gamma(\frac{1}{2} - \mathrm{i}p + \frac{\mathrm{i}}{2}q)\,(2\rho)^{2\mathrm{i}p}\, e^{-\pi p}\, \cosh \pi(p - \frac{1}{2}q)}{\Gamma(\frac{1}{2} + \mathrm{i}p + \frac{\mathrm{i}}{2}q)(e^{\pi q} + e^{-2\pi p})}\,. \tag{3.7}$$

Consequently, we find that the absorption probability for the extremal two-charge black hole in four dimensions is given by

$$P = 1 - |S_0|^2 = \frac{1 - e^{-2\pi q}}{1 + e^{-\pi(q+2p)}}\,, \tag{3.8}$$

when $\lambda_1\,\lambda_2 \ge (\ell + 1/2)^2$, while $P = 0$ otherwise. (In deriving these formulae, we have used the fact that the Gamma function obeys the identity $|\Gamma(1/2 + \mathrm{i}\,x)|^2 = \pi\,\mathrm{sech}(\pi\,x)$, where x is real.) In terms of the original dimensionless parameters $\lambda_i = \omega\,Q_i$, the absorption probability is therefore given by

$$P = \frac{1 - e^{-2\pi\sqrt{4\lambda_1\,\lambda_2 - (2\ell+1)^2}}}{1 + e^{-\pi(\lambda_1+\lambda_2+\sqrt{4\lambda_1\,\lambda_2-(2\ell+1)^2})}}\,, \qquad \lambda_1\,\lambda_2 \ge \Big(\ell + \frac{1}{2}\Big)^2\,, \tag{3.9}$$

with $P = 0$ if $\lambda_1\,\lambda_2 \le (\ell + 1/2)^2$.

Again, we may verify that the same result for the absorption probability is obtained by calculating the ratios of the ingoing fluxes at the horizon and at infinity. In this case, the flux is given by $F = \mathrm{i}\,\rho^2\,(\bar\phi\,\partial_\rho\phi - \phi\,\partial_\rho\bar\phi)$. From the results given above we find, after simple algebra, that the ingoing fluxes are given by

$$F_{\rm horizon} = \frac{|\alpha|^2\,e^{\pi q}\,\cosh\pi(p + \frac{1}{2}q)}{\sinh(\pi q)}\,, \qquad F_\infty = \frac{|\alpha|^2\,e^{\pi(p+q/2)}\,(e^{\pi q} + e^{-2\pi p})^2}{4\sinh^2(\pi q)}\,. \tag{3.10}$$

Taking the ratio $F_{\rm horizon}/F_\infty$, we indeed find that it agrees precisely with (3.9).

An alternative way of performing the calculation here is to redefine the notion of what constitutes an ingoing wave and what constitutes an outgoing wave. This can be done, for example, simply by defining the time dependence to be $e^{\mathrm{i}\omega t}$ rather than $e^{-\mathrm{i}\omega t}$. Since ω appears quadratically in the original wave equation, this means that we can simply choose universally to interpret the direction of motion of a wave front to be the opposite of the "conventional" choice. An alternative way of expressing this is that since the wave equation such as (3.1) for an energy eigenstate is real, one can complex conjugate the solution (3.3) and obtain another solution.

The upshot is that we can choose to reinterpret the requirement that the wave described by (3.3) be ingoing at the horizon as the requirement that the coefficient of $e^{-\mathrm{i}q/2}$ in (3.4) vanish, provided that we also define an ingoing wave at infinity to be one with $e^{\mathrm{i}\rho}$, as opposed to $e^{-\mathrm{i}\rho}$, dependence. If we do this, then the requirement that the wave on the horizon be ingoing becomes the condition $\alpha = 0$, so that the solution is

$$\phi = \beta\,\rho^{(\mathrm{i}q-1)/2}\,e^{-\mathrm{i}\rho}\,M\left(\frac{1}{2} + \mathrm{i}p + \frac{\mathrm{i}}{2}q, 1 + \mathrm{i}q, 2\mathrm{i}\rho\right)\,. \tag{3.11}$$

This is a simpler form for the solution than the one we discussed previously. At large ρ, (3.11) is proportional to $(-e^{\mathrm{i}\rho} + S_0\,e^{-\mathrm{i}\rho})$, with

$$S_0 = \frac{\mathrm{i}\,e^{-\pi q/2}\,(2\rho)^{-2\mathrm{i}\,p}\,\Gamma(\frac{1}{2} + \mathrm{i}\,p + \frac{\mathrm{i}}{2}\,q)}{\Gamma(\frac{1}{2} - \mathrm{i}\,p + \frac{\mathrm{i}}{2}\,q)}\,. \tag{3.12}$$

It is easy to see that this gives rise to the same result (3.8) for the absorption probability.

In [22], it was observed that the solution (3.11) could be expressed at large ρ in the form

$$\phi \sim \frac{1}{\rho}\,\sin\Big(\rho + p\,\log(2\rho) - \frac{1}{2}L\,\pi + \delta_{L}\Big)\,, \tag{3.13}$$

where $L = -1/2 + \mathrm{i}q/2\,q$ (we have adjusted the notation of [22] to fit in with ours). It follows from (3.12) that δ_{L} is a complex quantity, which we find to be

$$e^{2\mathrm{i}\,\delta_L} = \frac{\Gamma(\frac{1}{2} - \mathrm{i}\,p + \frac{\mathrm{i}}{2}\,q)}{\Gamma(\frac{1}{2} + \mathrm{i}\,p + \frac{\mathrm{i}}{2}\,q)}\,. \tag{3.14}$$

(Note that unlike the standard situation for Coulomb scattering, or single-charge black holes in $D = 4$, where δ_{L} is a real quantity given by the phase of the relevant Gamma function, here δ_{L} is not equal to the real quantity $\arg\Gamma(1/2 - \mathrm{i}\,p + \mathrm{i}q/2\,q)$, owing to the presence of the q term.) If the frequency in the ℓ'th mode is such that $\lambda_1\,\lambda_2 \le (\ell + 1/2)^2$, then both L and δ_{L} are real and there is no absorption, while if $\lambda_1\,\lambda_2 > (\ell + 1/2)^2$ they are both complex, and the absorption is given by (3.8), with p and q given by (3.2).

4. Geodesics and absorption probability

In the previous sections, we have obtained exact absorption probabilities in closed form for the extremal $D = 5$ single-charge and $D = 4$ two-charge black holes. The results are valid for massless scalar waves of arbitrary frequency. These two examples are of interest since such exact solutions for absorption probabilities are rare in gravity theories. In both cases, the results exhibit an interesting phenomenon, namely that the absorption probability is zero in a given mode if the frequency of the scalar wave is below some threshold value, related to the angular momentum ℓ of the mode. A related, but more extreme, situation also occurs in the $D = 4$ single-charge black hole, where the absorption probability vanishes for all values of the frequency of the scalar wave.

The vanishing of the absorption probabilities below certain threshold frequencies in the two cases studied in this paper is related to the behaviour of geodesics in these black holes. To see this, let us first consider radially-infalling timelike geodesics in the metric of an extremal N-charge p-brane in D dimensions. These are described by the lagrangean

$$L = \frac{1}{2}g_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} = -\frac{1}{2}\prod_\alpha H_\alpha^{-\frac{\tilde d}{D-2}}\,\Big(\frac{dt}{d\tau}\Big)^2 + \frac{1}{2}\prod_\alpha H_\alpha^{\frac{d}{D-2}}\,\Big(\frac{dr}{d\tau}\Big)^2\,, \tag{4.1}$$

where $d = p + 1$ and $\tilde d = D - d - 1$. The lagrangean takes the value $L = -1/2$ for a timelike geodesic and so this, together with the equation of motion for t, immediately gives us two first integrals:

$$\frac{dt}{d\tau} = E\prod_\alpha H^{\frac{\tilde d}{D-2}}\,,$$
$$\Big(\frac{dr}{d\tau}\Big)^2 = E^2\prod_\alpha H^{\frac{\tilde d - d}{D-2}} - \prod_\alpha H^{-\frac{d}{D-2}}\,, \tag{4.2}$$

where E is a constant of integration. Thus near the horizon at $r=0$, we have that $H_\alpha \sim r^{-\tilde d}$, and hence

$$\frac{dr}{dt} = \frac{dr}{d\tau}\,\frac{d\tau}{dt} \sim -r^{\frac12 N\,\tilde d}\,. \tag{4.3}$$

Thus we see that the coordinate time t taken for the geodesic to reach the horizon is either finite, logarithmically-divergent, or power-law divergent as a function of r, according to the following inequality:

$$\begin{aligned}
&\frac12 N\,\tilde d < 1: &&\text{finite coordinate time}\\
&\frac12 N\,\tilde d = 1: &&\text{logarithmically-divergent coordinate time}\\
&\frac12 N\,\tilde d > 1: &&\text{power-law-divergent coordinate time}
\end{aligned} \tag{4.4}$$

It is straightforward to verify that the same conclusions hold for null geodesics. (Note that in the case of black holes we have $\tilde d = D-3$, but that the conditions (4.4) apply equally to any extremal N-charge p-branes.) Thus we see that amongst the single-charge black holes, the case $D=4$ is distinguished by the fact that timelike or null geodesics reach the horizon in a finite coordinate time. (See, for example, [23].) In $D=5$, the time taken is logarithmically divergent, whilst in $D\ge 6$ it is power-law divergent. The 2-charge black hole in $D=4$ again has a logarithmically-divergent coordinate time, while for all $N\ge 3$ charge black holes in $D\ge 4$, and all $N\ge 2$ charge black holes in $D\ge 5$, it again takes a power-law-divergent coordinate time to reach the horizon.

We therefore see the following correspondences. If a timelike or null geodesic reaches the horizon in finite coordinate time, there is no absorption at all. If it takes a logarithmically-divergent coordinate time, there is absorption provided that the frequency of the ingoing wave exceeds some finite bound, $\omega > \omega_0$; this is the situation for the $D=5$ single-charge, and $D=4$ two-charge black holes that we have considered in this paper. Finally, if a timelike geodesic reaches the horizon in a power-law-divergent coordinate time, there is non-vanishing absorption for all frequencies $\omega > 0$.

The correspondence between the behaviour of infalling geodesics and the absorption probability may not be too surprising, in view of the fact that the radial coordinate-dependence of the scalar wave function near the horizon of a p-brane is closely related to the dependence of the radial coordinate of an infalling geodesic on the coordinate time t. For example, the coordinate time t depends logarithmically on the radial coordinate r of an infalling geodesic in the $D = 5$ single-charge and $D = 4$ two-charge black holes. A similar dependence, namely $\phi \sim e^{ik \log \rho}$, appears in the near-horizon form of the solutions (2.4), (3.4) of the wave equation in the corresponding black-hole backgrounds. More generally, for any extremal p-brane the wave equation is given by (1.2), and the solution near the horizon at $\rho = 0$ is given by (5.11). The near-horizon solution is therefore wavelike if $\mu \geq 0$, but instead has just a power-law dependence on ρ if $\mu < 0$. The sign of μ, which is given by (5.8), is thus precisely correlated with the cases enumerated in (4.4).

5. Leading order absorption by extremal p-branes

In general the wave equation (1.2) cannot be solved analytically. One approach is to solve the equation in two distinct but overlapping regions, namely an inner region near to the p-brane, and an outer region reaching to infinity. Such a technique has been widely used recently in obtaining low energy absorption sections for various black holes and p-branes [1, 2, 3, 4, 5, 8, 9, 6, 7, 10, 11, 12, 14, 15].

For the outer solution, it is convenient to define $\phi = \rho^{-(\tilde{d}+1)/2}\, \psi$, leading to the wave equation

$$\psi'' + \left[\frac{1-(2\ell+\tilde{d})^2}{4\rho^2} + \prod_{\alpha=1}^{N}\left(1+\frac{\lambda_\alpha^{\tilde{d}}}{\rho^{\tilde{d}}}\right)\right]\psi = 0\,. \tag{5.1}$$

For notational convenience, we define

$$\Lambda = \left(\prod_{\alpha=1}^{N} \lambda_\alpha\right)^{1/N}\,. \tag{5.2}$$

(This means that if all the charges are equal we have $\lambda_\alpha = \Lambda$.) In the region where

$$\rho \gg \Lambda^{N\tilde{d}/(N\tilde{d}-2)}\,, \tag{5.3}$$

the equation (5.1) can be approximated as

$$\psi'' + \left[\frac{1-(2\ell+\tilde{d})^2}{4\rho^2} + 1\right]\psi = 0\,, \tag{5.4}$$

which can be solved in terms of Bessel functions. Thus the outer solution is given by

$$\phi_{\rm outer} = \rho^{-\tilde{d}/2}\left(\alpha\, J_{\ell+\tilde{d}/2}(\rho) + \beta\, Y_{\ell+\tilde{d}/2}(\rho)\right)\,. \tag{5.5}$$

Note that the constraint (5.3) is derived with the assumption that $\tilde{d} > 2$. Otherwise the constraint would be $\rho \gg \Lambda$. Thus the cases with $\tilde{d} = 2$ and 1 are exceptional; these were discussed in the previous sections.

In the inner region, the constant 1 in the harmonic function can be dropped, provided that[1]

$$\rho \ll \Lambda\,. \tag{5.6}$$

It is convenient to use a new variable z, and to define a new wavefunction, as follows:

$$z = \frac{\Lambda^{N\tilde{d}/2}}{u\,\rho^u}\,, \qquad \phi = z^{\frac{1}{2}(v-1)}\, f(z)\,, \tag{5.7}$$

where

$$u = \frac{N\tilde{d}-2}{2}\,, \qquad v = \frac{2\tilde{d}}{N\tilde{d}-2}\,. \tag{5.8}$$

The equation (1.2) then becomes

$$f'' + \left[\frac{1-(2L+v)^2}{4\rho^2} + \prod_{\alpha=1}^{N}\left(1+\left(\frac{\Lambda}{\lambda_\alpha}\right)^{\tilde{d}}\frac{(\frac{\Lambda}{u})^v}{z^v}\right)\right] f = 0\,. \tag{5.9}$$

where $L = 2\ell/(N\tilde{d}-2)$. Specifically, when $N=1$ we have $v>2$, and hence the limit where the approximation to (5.9) is a good one is when $z \gg \Lambda^{N\,v/(N\,v-2)}$, which implies $\rho \ll 1$. If, on the other hand we have $N>1$, then $v<2$, and hence the approximation is good when $z \gg \Lambda$, implying (5.6). If we make this approximation then equation (5.9) becomes

$$f'' + \left[\frac{1-(2L+v)^2}{4z^2} + 1\right] f = 0\,. \tag{5.10}$$

which is solvable in terms of Bessel functions, giving

$$\begin{aligned}\phi_{\rm inner} &= z^{v/2}\,(c_1\,J_{L+v/2}(z) + c_2\,Y_{L+v/2}(z))\\ &= \frac{\Lambda^{vN\tilde{d}/4}}{u^{v/2}\,\rho^{\tilde{d}/2}}\left[c_1\,J_{L+v/2}\left(\frac{\Lambda^{N\tilde{d}/2}}{u\,\rho^u}\right) + c_2\,Y_{L+v/2}\left(\frac{\Lambda^{N\tilde{d}/2}}{u\,\rho^u}\right)\right]\,.\end{aligned} \tag{5.11}$$

In order for the solution to describe a purely ingoing wave near the horizon at $\rho=0$, we see from (A.1) that we must have $c_2 = {\rm i}\,c_1$. The ingoing wave on the horizon is then given by $\phi \sim -{\rm i}\,c_2\,\sqrt{2/\pi}\,z^{(v-1)/2}\,e^{{\rm i}\,z}\,e^{-{\rm i}\,\pi(2L+v+1)/4}$.

The inner and outer solutions are valid in the regions specified in (5.6) and (5.3), and thus it follows that in order to have an overlap where there is a range of common validity for the two solutions, the frequency ω must be sufficiently small that

$$\frac{\Lambda^{N\,v}}{\Lambda} = \Lambda^{1/u} \ll 1 \tag{5.12}$$

if $N>1$, while instead $\Lambda^{N\,v} \ll 1$ if $N=1$. This implies that the approximations are valid in an appropriately low-energy regime. In such a regime, both the coordinates

[1]In the case of $N=1$, the valid region can actually be $\rho \ll 1$.

z and ρ, appearing respectively in the Bessel functions in the inner and outer regions, are much less than 1, and so we can perform small-argument series expansions on the Bessel functions in both regions, and hence we can match the two solutions in the leading orders. In the inner region the boundary condition on the horizon has already determined that the integration constants c_1 and c_2 are of equal modulus, since $c_2 = \mathrm{i}\, c_1$, but in the outer region the relative sizes of the integration constants α and β are yet to be determined. However, since the structure in the inner region is established, it is a simple matter to recognise the most dominant terms in the small-z power-series expansion of $\phi_{\rm inner}$, and then to require that these match with the small-ρ power-series expansion of $\phi_{\rm outer}$. Note that we need to match both the functions and their derivatives in the overlap region, implying that the functional forms of the inner and outer solutions at leading order must be identical. Structurally, these have the following forms:

$$\begin{aligned}
\text{Inner}: \phi &\sim c_1\, z^{v+L} + c_2\, (z^{-L} + z^{-L+2})\,,\\
&\sim c_1\, \rho^{-\tilde{d}-\ell} + c_2\, (\rho^{\ell} + \rho^{\ell+2u})\,,\\
\text{Outer}: \phi &\sim \alpha(\rho^{\ell} + \rho^{\ell+2}) + \beta\, \rho^{-\tilde{d}-\ell}\,.
\end{aligned} \tag{5.13}$$

It is straightforward to see that z^{-L}, (or equivalently ρ^{ℓ}), is the leading order of the expansion, since $z \ll 1$ in the overlapping region. For generic value of $L = 2\ell/(N\tilde{d}-2)$, the subleading order term is z^{-L+2}. It follows that in general we can only match the leading order term, providing a determination of α, but not β. However, in some special cases, namely $v + L < 2 - L$, i.e.

$$\frac{2\ell + \tilde{d}}{N\tilde{d} - 2} < 1\,, \tag{5.14}$$

the subleading order term is z^{v+L}. In these cases, we can match the inner and outer solutions at the level both of the leading constant order and also the first sub-leading order $\rho^{-\tilde{d}}$. Equating the coefficients of ρ^{ℓ} and $\rho^{-\tilde{d}-\ell}$ in the overlap region thus enables us to obtain leading-order expressions for both the integration constants α and β in the outer solution. Note that for $\ell = 0$, the condition (5.14) is satisfied provided that $N \ge 2$.

There are two methods available for calculating the absorption probability. The first is by calculating the ratio of the ingoing flux at the horizon, divided by the ingoing flux at infinity. This method does not require the knowledge of the constant β. The second is by calculating the S-matrix, which is governed by the ratio of β to α. This second method can be used only if β, in addition to α, is known. For now, we shall therefore employ the flux-ratio method.

As we saw above, to determine α we need only to keep the leading order in a power-series expansion. We find from (5.11) and (A.2) that this is given by

$$\phi_{\rm inner} = -\frac{c_2}{\pi}\, 2^{L+v/2}\, \frac{u^L}{\Lambda^{N\tilde{d}L/2}}\, \Gamma(L + v/2)\, \rho^{\ell} + \cdots\,. \tag{5.15}$$

On the other hand for the outer solution, we find from (5.5) that in the small-argument expansion appropriate to the overlap region the leading-order term is given by

$$\phi_{\rm outer} \sim \frac{\alpha\, 2^{-\ell-\tilde d/2}}{\Gamma(\ell+1+\frac12\tilde d)}\, \rho^\ell + \cdots\,. \tag{5.16}$$

(The β coefficient in the outer solution is negligible, at this leading order, in comparison to α.) Equating the expressions in (5.15) and (5.16), we obtain the relation between the coefficients α and c_2 of the outer and inner solutions. The absorption coefficient is most easily calculated at this leading order as the ratio of the ingoing flux at the horizon, divided by the ingoing flux at infinity. In general, this flux may be defined as

$$F = {\rm i}\, \rho^{\tilde d+1}\, \Big(\bar\phi\, \frac{\partial\phi}{\partial\rho} - \phi\, \frac{\partial\bar\phi}{\partial\rho}\Big)\,, \tag{5.17}$$

where ϕ here is taken to be the ingoing component of the wave. From the asymptotic forms of $\phi_{\rm inner}$ and $\phi_{\rm outer}$ where the arguments of the Bessel functions are large, we find from (5.11), (5.5) and (A.1) that the ingoing fluxes at the horizon and at infinity are given by

$$F_{\rm horizon} = \frac{4}{\pi}\, |c_2|^2\, u^{1-v}\, \Lambda^{N\tilde d\, v/2}\,,\qquad F_\infty = \frac{|\alpha|^2}{\pi}\,, \tag{5.18}$$

and hence to leading order the absorption probability $P\equiv F_{\rm horizon}/F_\infty$ is

$$P_\ell = \frac{2\pi^2\, (2u)^{1-v-2L}\, \Lambda^{N\tilde d(L+v/2)}}{2^{2\ell+\tilde d}\, \Gamma(\ell+1+\frac12\tilde d)^2\, \Gamma(L+\frac12 v)^2}\,. \tag{5.19}$$

Finally, we note that the phase-space factor relating the absorption probability to the scattering cross-section σ is [24]

$$\sigma_\ell = 2^{\tilde d}\, \pi^{\tilde d/2}\, \Gamma\Big(\frac12\tilde d\Big)\, \Gamma\Big(\ell+1+\frac12\tilde d\Big)\, \binom{\ell+\tilde d-1}{\ell}\, \omega^{-2\ell-\tilde d-1}\, P_\ell\,. \tag{5.20}$$

Hence we arrive at the result for the scattering cross-section

$$\sigma_\ell = \frac{2\pi^{2+\tilde d/2}\, (2u)^{1-v-2L}\, \Gamma(\frac{\tilde d}{2})\, \Lambda^{N\tilde d(L+\frac12 v)}\, \omega^{-\tilde d-1}}{2^{2\ell}\, \Gamma(\ell+1+\frac12\tilde d)\, \Gamma(L+\frac12 v)^2}\, \binom{\ell+\tilde d-1}{\ell}\,. \tag{5.21}$$

The above result is the leading-order contribution to the low-energy absorption. In this case the $\ell=0$ s-wave absorption is dominant, and is given by

$$\sigma_0 = \frac{2\pi^{2+\tilde d/2}\, (2u)^{1-v}\, \Gamma(\frac12\tilde d)\, (Q_1 Q_2\cdots Q_n)^{v/2}\omega^{v-1}}{\Gamma(1+\tilde d)\, \Gamma(\frac12 v)^2}\,. \tag{5.22}$$

Thus at low energy the absorption cross-section σ of a generic extremal p-brane, and the frequency of the incoming wave, are related by

$$\sigma = c\, (Q_1\, Q_2\, \cdots\, Q_N)^{\frac{\tilde d}{N\tilde d-2}}\, \omega^{\frac{2\tilde d}{N\tilde d-2}-1}\,, \tag{5.23}$$

where c is some purely numerical constant. It should be recalled that the above result is applicable only when $N\tilde{d} - 2 > 0$. If instead $N\tilde{d} - 2 \le 0$, which arises for $D = 5$ single-charge and $D = 4$ single or two-charge black holes, the absorption probabilities vanish at the low energies, below certain thresholds. The closed-form results in these cases were obtained in sections 2 and 3. Note that when the condition (5.14) is satisfied, (for example, $\ell = 0$ and $N \ge 2$) both α and β in (5.5) can be determined by matching the inner and outer solutions in the overlap region. In this case, the absorption can also be calculated by the S-matrix method.

In the next section we shall show that the leading-order relationship (6.8) between the cross-section and the frequency coincides with the leading-order relation between the entropy and the temperature of the corresponding near-extremal p-brane.

Finally, in this section, we note that the original wave equation (5.1), and the transformed wave equation (5.9), are identical in form if we make the replacements

$$\tilde{d} \longleftrightarrow v \qquad \ell \longleftrightarrow L\,, \qquad \lambda_\alpha \longleftrightarrow \frac{\Lambda}{u}\left(\frac{\Lambda}{\lambda_\alpha}\right)^{\tilde{d}/v}\,. \tag{5.24}$$

In cases where v is an integer, and furthermore is equal to d, then $(\tilde{d}, d)$ form a dual pair, associated with $(d-1)$-branes and $(\tilde{d}-1)$-branes in $D = d + \tilde{d} + 2$ dimensions. When such a dual pair arises, the dilaton becomes constant near the horizon of the p-brane. In cases such as the D3-brane, the dyonic string or the $D = 4$ four-charge black hole, we have $\tilde{d} = d$ and $L = \ell$, and in these cases it follows that there is an exact duality symmetry of the wave equation in the inner region and the outer region, for arbitrary partial waves. On the other hand for M2-brane and M5-brane, the duality that maps between the inner and outer solutions is exact only for the s-wave, as was observed in [10]. For example, the outer solution in the M2-brane with partial wave-number ℓ is mapped to the inner solution in the M5-brane with partial wave-number $\ell/2$. The case for the three-charge $D = 5$ black-hole/string duality is analogous. The outer solution in the black hole background with partial wave-number ℓ is mapped to the inner solution in the string background with partial wave-number $\ell/2$. This occurrence of half-integers may be indicative of an intrinsic rôle for fermions in this discussion of duality.

6. $S(T)$ versus $\sigma(\omega)$

The extremal p-brane solutions (1.1) can easily be generalised to non-extremal ones, with [25, 26]

$$\begin{aligned} ds^2 &= \prod_{\alpha=1}^{N} H_\alpha^{-\frac{\tilde{d}}{D-2}}\,(-e^{2f}\,dt^2 + dx^i\,dx^i) + \prod_{\alpha=1}^{N} H_\alpha^{\frac{d}{D-2}}\,(e^{-2f}\,dt^2 + r^2\,d\Omega^2)\,, \\ e^{2f} &= 1 - \frac{k}{r^{\tilde{d}}}\,, \qquad H_\alpha = 1 + \frac{k}{r^{\tilde{d}}}\sinh^2\mu_\alpha\,. \end{aligned} \tag{6.1}$$

The N charges Q_α are given in terms of the parameters k and μ_α by

$$Q_\alpha = \frac{1}{2} k \, \sinh 2\mu_\alpha \,. \tag{6.2}$$

The extremal limit is obtained by sending $k \longrightarrow 0$ and $\mu_\alpha \longrightarrow \infty$, while keeping the Q_α fixed.

The outer horizon of the metric (6.1) is located at $r_+ = k^{1/\tilde{d}}$. It straightforward to show that the Hawking temperature and entropy are given by

$$T = \frac{\tilde{d}}{4\pi r_+} \prod_{\alpha=1}^{N} (\cosh \mu_\alpha)^{-1} \,, \qquad S = \frac{1}{4} r_+^{\tilde{d}+1} \, \Omega_{\tilde{d}+1} \prod_{\alpha=1}^{N} \cosh \mu_\alpha \,, \tag{6.3}$$

where Ω_n is the volume of the unit n-sphere. For convenience, we may define a "scaled" entropy $\widetilde{S} \equiv 16\pi/(\tilde{d}\, \Omega_{\tilde{d}+1})\, S$, in terms of which we have

$$\widetilde{S}\, T = k \,. \tag{6.4}$$

From the expression (6.2) for the charges, we see that

$$\cosh \mu_\alpha = \sqrt{\frac{k + \sqrt{k^2 + 4Q_\alpha^2}}{2k}} \,. \tag{6.5}$$

From this, it follows that

$$\begin{aligned} \widetilde{S} &= \frac{4\pi}{\tilde{d}}\, k^{1-1/v} \prod_{\alpha=1}^{N} \sqrt{\frac{1}{2} k + \sqrt{\frac{1}{4} k^2 + Q_\alpha^2}} \\ &= \frac{4\pi}{\tilde{d}}\, (\widetilde{S}\, T)^{1-1/v} \prod_{\alpha=1}^{N} \sqrt{\frac{1}{2} (\widetilde{S}\, T) + \sqrt{\frac{1}{4} (\widetilde{S}\, T)^2 + Q_\alpha^2}} \,, \end{aligned} \tag{6.6}$$

where v is defined in (5.8). Hence we obtain an implicit relationship between entropy and temperature, given by

$$\widetilde{S} = \tilde{c}\, T^{v-1} \prod_{\alpha=1}^{N} \left(\frac{1}{2} (\widetilde{S}\, T) + \sqrt{\frac{1}{4} (\widetilde{S}\, T)^2 + Q_\alpha^2} \right)^{v/2} \,, \tag{6.7}$$

where we have defined $\tilde{c} = (4\pi/\tilde{d})^v$.

In the near-extremal regime, where $\widetilde{S}\, T = k \ll Q_\alpha$, we can therefore obtain the leading-order expansion of entropy in terms of temperature, given by

$$\widetilde{S}_0 = \tilde{c}\, (Q_1\, Q_2 \, \cdots \, Q_N)^{\frac{\tilde{d}}{N\tilde{d}-2}} \; T^{\frac{2\tilde{d}}{N\tilde{d}-2} - 1} \,. \tag{6.8}$$

Thus the leading-order relationship between the entropy and temperature of a near-extremal p-brane coincides precisely with the leading-order relationship (5.23) between the scattering cross-section and the frequency for scalar s-waves in the corresponding extremal p-brane. (Such a correspondence between $S(T)$ and $\sigma(\omega)$ breaks down when $N\tilde{d} - 2 \le 2$, which is associated with finite or logarithmically-divergent coordinate time for geodesics to reach the horizon. In these cases, the temperature increases as the p-brane approaches extremality. When $N\tilde{d} = 2$, the temperature reaches a finite maximum at the extremal limit, whilst for $N\tilde{d} < 2$, the temperature goes to infinity at the extremal limit. Such black holes were discussed in [27], and they can be viewed as elementary particles.)

It is of interest to look at the higher-order terms in the expansion for the entropy as a function of temperature. Note that the expansion parameter is $k = (\widetilde{S}_0\, T) \sim T^v$. The expansion can easily be obtained from the implicit expression for S in terms of T given in (6.7), by means of the following iterative algorithm. If we define $\widetilde{S}_i$ to be the entropy accurate up to order $T^{(i+1)v-1}$, then by substuting $\widetilde{S}_i$ into the right-hand side of equation (6.7), we obtain a new expression $\widetilde{S}_{i+1}$ that is accurate up to order $T^{(i+2)v-1}$:

$$\widetilde{S}_{i+1} = \tilde{c}\, T^{v-1} \prod_{\alpha=1}^{N} \left(\frac{1}{2}(\widetilde{S}_i\, T) + \sqrt{\frac{1}{4}(\widetilde{S}_i\, T)^2 + Q_\alpha^2} \right)^{v/2}, \tag{6.9}$$

With the leading order $\widetilde{S}_0$ given by (6.8), we can then obtain the relation between the entropy and the temperature to arbitrary order, as a power series in T^v. We find that the first few orders are

$$\begin{aligned} \frac{S}{S_0} &= 1 + \frac{1}{4}\xi_1\, T^v + \frac{3}{32}\xi_1^2\, T^{2v} + \frac{1}{96}(4\xi_1^3 - \xi_3)\, T^{3v} + \frac{5}{6144}(25\xi_1^4 - 16\xi_1\,\xi_3)\, T^{4v} + \\ &\quad + \frac{3}{2560}(\xi_1^5 - 10\xi_1^2\,\xi_3 + \xi_5)\, T^{5v} + \mathcal{O}(T^{6v}) \,, \end{aligned} \tag{6.10}$$

where

$$\xi_n \equiv \tilde{c}\, v^n \left(\prod_{\alpha=1}^{N} Q_\alpha \right)^{n\,v/2} \sum_{\alpha=1}^{N} \frac{1}{(Q_\alpha)^n} \,. \tag{6.11}$$

Note that the first two sub-leading terms depend on the charges only through the function ξ_1, but that a new structure ξ_3, with different functional dependence on the charges, emerges at the third order in the expansion. Similarly, yet another function of the charges, ξ_5, emerges at the fifth order, and so on. In general, the expansion (6.10) up to order $T^{m\,v}$ will involve the q independent functions $\{\xi_1, \xi_3, \xi_5, \ldots, \xi_{2q-1}\}$, where $q = [(m+1)/2]$ is the integer part of $(m+1)/2$.

Having shown that the leading-order relationship between entropy and temperature for near-extremal p-branes coincides with the leading order in the relationship between the scattering cross-section and the frequency for low-frequency scalar s-waves in the corresponding extremal p-branes, it is of interest to investigate how far this parallel extends at higer orders in the expansion. There are only a limited number of cases where higher-order corrections to the scattering cross-sections have been reliably obtained. One of these is the D3-brane, for which the first few orders are [13]

$$\frac{\sigma}{\sigma_0} = 1 - \frac{1}{6} Q\, \omega^4\, \log(e^\gamma\, Q^{1/4}\, \omega) + \frac{7}{72}\, Q\, \omega^4 + \cdots \tag{6.12}$$

where γ is the Euler's constant. On the other hand, the relation between the entropy and temperature for the near-extremal D3-brane can be seen from (6.10) to be given by

$$\frac{S}{S_0} = 1 + 4\pi^4\, Q\, T^4 + \cdots\,. \tag{6.13}$$

Another example is the dyonic string in $D = 6$, for which the first few terms in the expansion for the s-wave scattering cross-section are [16]

$$\frac{\sigma}{\sigma_0} = 1 - 2(Q_e + Q_m)\,\omega^2 \log(e^\gamma\,\lambda) + (Q_e + Q_m)\,\omega^2 + \cdots\,. \tag{6.14}$$

Here we have defined $\lambda^2 = \sqrt{Q_e\, Q_m}\,\omega^2/4$. By contrast, the relation between the entropy and temperature, which can be read off from (6.10), is

$$\frac{S}{S_0} = 1 + 8\pi^2\,(Q_e + Q_m)\,T^2 + \cdots\,. \tag{6.15}$$

Thus we see that in these two examples there are additional terms, depending upon the logarithm of the frequency, in the relation between the scattering cross-section and the frequency.

A further example is the 3-charge black hole in $D = 5$. The low energy absorption cross section of the extremal black hole was obtained in [3], in the case where the charge Q_3 is much smaller than the other two charges Q_1 and Q_2. It is given by

$$\frac{\sigma}{\sigma_0} = 1 + \frac{1}{4}\pi\,\left(\frac{Q_1\,Q_2}{Q_3}\right)^{1/2}\omega + \frac{1}{48}\pi^2\,\frac{Q_1\,Q_2}{Q_3}\,\omega^2 + \mathcal{O}(\omega^4)\,. \tag{6.16}$$

The corresponding expression for the entropy in terms of the temperature can be read off from (6.10). For the case where $Q_3 \ll Q_1, Q_2$, we find that it is given by

$$\frac{S}{S_0} = 1 + \pi\,\left(\frac{Q_1\,Q_2}{Q_3}\right)^{1/2} T + \frac{3}{2}\pi^2\,\frac{Q_1\,Q_2}{Q_3} T^2 + 2\pi^3\,\left(\frac{Q_1\,Q_2}{Q_3}\right)^{3/2} T^3 + \mathcal{O}(T^4)\,. \tag{6.17}$$

Thus we see that the expansion parameters are exactly the same, although the precise numerical coefficients disagree.

A. Asymptotic forms of Bessel and Kummer functions

The asymptotic expansions of the Bessel functions for large values of their arguments are

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{\pi\,x}}\,\cos\left(x - \frac{1}{2}\nu\,\pi - \frac{1}{4}\pi\right)\,, \\ Y_\nu(x) &\sim \sqrt{\frac{2}{\pi\,x}}\,\sin\left(x - \frac{1}{2}\nu\,\pi - \frac{1}{4}\pi\right)\,. \end{aligned} \tag{A.1}$$

For small arguments, the Bessel functions can be approximated by

$$\begin{aligned} J_\nu(x) &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu} - \frac{1}{\Gamma(\nu+2)}\left(\frac{x}{2}\right)^{\nu+2} + \cdots \,, \\ Y_\nu(x) &= -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu} + \cdots \,. \end{aligned} \tag{A.2}$$

Kummer's confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$ satisfy the differential equation $z\,w'' + (b-z)\,w' - a\,w = 0$. The solution $M(a,b,z)$ is known as the regular solution, and it has the power-series expansion

$$\begin{aligned} M(a,b,z) &= \sum_{n\geq 0} \frac{(a)_n\, z^n}{(b)_n\, n!} \\ &= 1 + \frac{a\,z}{b} + \frac{a(a+1)\,z^2}{2!\,b(b+1)} + \frac{a(a+1)(a+2)\,z^3}{3!\,b(b+1)(b+2)} + \cdots \,, \end{aligned} \tag{A.3}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the ascending Pochhammer symbol. The irregular function $U(a,b,z)$ is expressible in terms of $M(a,b,z)$ as follows:

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\,\Gamma(b)} - z^{1-b}\,\frac{M(1+a-b,2-b,z)}{\Gamma(a)\,\Gamma(2-b)}\right]. \tag{A.4}$$

This, together with (A.3), can be used to determine the small-$|z|$ dependence of the confluent hypergeometric functions.

The asymptotic behaviour of the confluent hypergeometric functions at large $|z|$ is as follows:

$$\begin{aligned} U(a,b,z) &\sim z^{-a}\left(1 + O(z^{-1})\right)\,, \\ M(a,b,z) &\sim \frac{\Gamma(b)\, e^{\pm i\pi a}\, z^{-a}}{\Gamma(b-a)}\left(1 + O(z^{-1})\right) + \frac{\Gamma(b)\, e^{z}\, z^{a-b}}{\Gamma(a)}\left(1 + O(z^{-1})\right)\,, \end{aligned} \tag{A.5}$$

where in the second line the upper sign is taken if $-\pi/2 < \arg z < 3\pi/2$, whilst the lower sign is taken if $-3\pi/2 < \arg z \leq -\pi/2\pi$.

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